# On the Density of Integers $n$ Divisible by a Certain Integer $m$ Such That $m$ Does Not Divide $\sigma(n)$-$(n)$



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## 1 Introduction

The purpose of this paper is to present the proof of a theorem concerning the density of integers $n$ divisible by a certain integer $m$ such that $m$ doesn't divide $\sigma(n)$-$(n)$. We will demonstrate that this density is zero by giving an asymptotic upper bound of the cardinality of these numbers $n$ less than a real $x$.

## 2 Notations and Definitions

In the rest of the paper:

Let $m$ be a fixed natural integer $\geq 3$.

$x$ and $t$ will designate positive real numbers.

We denote (mod $m$) by $[m]$ for brevity.

We define the sum-of-divisors and sum-of-proper-divisors functions like so:

$$\sigma(n) = \sum_{d|k} d. \tag{1}$$

And

$$\sigma'(n) = \sigma(n) - n. \tag{2}$$

And we also define the function $\phi(n)$ as Euler's totient function, which counts the number of integers $\leq n$ that are coprime to $n$.

## 3 Theorem 1

The (asymptotic) density of the integers $n$ divisible by $m$ such that $m$ doesn't divide $\sigma'(n)$ is zero.

Furthermore, we also give the following asymptotic upper bound for any sufficiently large real number x:

$$A_m(x) := card\{n \leq x \ such \ that \ m|n \ and \ m \not| \sigma'(n)\} = O\left(\frac{x}{(\ln \ln x)^{\frac{1}{\phi(m)}}}\right). \quad (3)$$

## 4 Proof of the Theorem

In order to prove this theorem, we need to prove the following intermediate lemma:

## 5 Lemma

For all real numbers $x$:

$$S_m(x) := card\{n \leq x \ such \ that \ m \not| \sigma(n)\} = O\left(\frac{x}{(\ln \ln x)^{\frac{1}{\phi(m)}}}\right). \quad (4)$$

## 6 Proof of the Lemma

Let $x$ and $t$ be two sufficiently large reals such that $1 \ll t \ll x$.

It's clear that if a prime number $q$ exists such that $q \equiv -1[m]$, $q|n$, and $q^2 \not| n$, then $m|\sigma(n)$.

Let $q_1 < q_2 < ...$ be the prime numbers such that $q_i \equiv -1[m]$.

From a corollary of a theorem by Dirichlet concerning the prime numbers in arithmetic progressions, we have the following:

$$\sum_{q\leq t, q\equiv -1[m]} \frac{1}{q} = \frac{\ln\ln t}{\phi(m)} + O(1)\,. \tag{5}$$

And from this we deduce that:

$$\sum_{q_i\leq t, q_i\equiv -1[m]} \frac{q_i-1}{q_i^2} = \frac{\ln\ln t}{\phi(m)} + O(1)\,. \tag{6}$$

Now we will consider the following product: $\prod_{q_i\leq t, q_i\equiv -1[m]}(1-\frac{q_i-1}{q_i^2})$
We know that for all real $y$ such that $0\leq y<1$, the following inequality holds:

$$\ln(1-y)\leq -y. \tag{7}$$

Thus:

$$\ln(\prod_{q_i\leq t, q_i\equiv -1[m]}(1-\frac{q_i-1}{q_i^2}))\leq -(\sum_{q_i\leq t, q_i\equiv -1[m]}\frac{q_i-1}{q_i^2}) \tag{8}$$

Whence:

$$\ln(\prod_{q_i\leq t, q_i\equiv -1[m]}(1-\frac{q_i-1}{q_i^2}))\leq -(\frac{\ln\ln t}{\phi(m)}) + O(1) \tag{9}$$

And from this we deduce that:

$$\prod_{q_i\leq t, q_i\equiv -1[m]}(1-\frac{q_i-1}{q_i^2})) = O\left(\exp(\frac{-\ln\ln t}{\phi(m)})\right) = O\left(\frac{1}{(\ln t)^{\frac{1}{\phi(m)}}}\right) \tag{10}$$

Now we'll consider the following product:

$$Q_t = \prod_{q_i\leq t, q_i\equiv -1[m]} q_i \tag{11}$$

If for a certain integer $a$ such that $1\leq a\leq Q_t^2$ and for a prime number $q_i$ ($q_i\leq t, q_i\equiv -1[m]$) such that $q_i|a$ and $q_i^2\not| a$ and $n\equiv a[m]$, then $m|\sigma(n)$.
By applying the sieve of Eratosthenes, we'll find that the number of classes of

residues (mod $Q_t^2$) for which the preceding property isn't checked for any index $i$ is equal to:

$$Q_t^2 \prod_{q_i \leq t, q_i \equiv -1[m]} (1 - \frac{1}{q_i} + \frac{1}{q_i^2}) = Q_t^2 \prod_{q_i \leq t, q_i \equiv -1[m]} (1 - \frac{q_i - 1}{q_i^2}) \tag{12}$$

From this we thus conclude:

$$S_m(x) := card\{n \leq x \ such \ that \ m \not| \sigma(n)\} \leq (\frac{x}{Q_t^2} + 1) Q_t^2 \prod_{q_i \leq t, q_i \equiv -1[m]} (1 - \frac{q_i - 1}{q_i^2}). \tag{13}$$

Whence:

$$S_m(x) \leq x \prod_{q_i \leq t, q_i \equiv -1[m]} (1 - \frac{q_i - 1}{q_i^2}) + Q_t^2 \prod_{q_i \leq t, q_i \equiv -1[m]} (1 - \frac{q_i - 1}{q_i^2}). \tag{14}$$

If we define $t = \frac{\ln x}{2}$, the theorem of primes in arithmetic progressions shows that:

$$\ln(Q_t) \sim \frac{\ln x}{2\phi(m)} \tag{15}$$

Whence:

$$\ln(Q_t^2) \sim \frac{\ln x}{\phi(m)} \tag{16}$$

And since $2 \leq \phi(m)$ because $3 \leq m$, then:

$$Q_t^2 = o(x) \tag{17}$$

Thus:

$$S_m(x) = O\left( x \prod_{q_i \leq t, q_i \equiv -1[m]} (1 - \frac{q_i - 1}{q_i^2}) \right). \tag{18}$$

Whence the following result:

$$S_m(x) = O\left( \frac{x}{(\ln \ln x + \ln(\frac{1}{2}))^{\frac{1}{\phi(m)}}} \right) = O\left( \frac{x}{(\ln \ln x)^{\frac{1}{\phi(m)}}} \right) \tag{19}$$

QED.

# 7 Following the Proof of the Theorem

Now we'll use the lemma and start by remarking that if $m|n$ and $m\not|\sigma'(n)$ then $m\not|\sigma(n)$.

Thus:

$$\{n \leq x \text{ such that } m|n \text{ and } m\not|\sigma'(n)\} \subset \{n \leq x \text{ such that } m\not|\sigma(n)\}. \tag{20}$$

Whence:

$$A_m(x) := card\{n \leq x \text{ such that } m|n \text{ and } m\not|\sigma'(n)\} \leq S_m(x) := card\{n \leq x \text{ such that } m\not|\sigma(n)\} \tag{21}$$

And from the lemma we deduce:

$$A_m(x) := card\{n \leq x \text{ such that } m|n \text{ and } m\not|\sigma'(n)\} = O\left(\frac{x}{(\ln\ln x)^{\frac{1}{\phi(m)}}}\right) \tag{22}$$